An M/G/1 Queue With Exceptional Service For The First N Customer In Each Busy Period And Server Vacation

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Abstract:

We consider a single server queueing system where the first N customer of each busy period receives exceptional service, in addition the server takes single vacation each time the system becomes empty. Using supplementary variable approach, we obtain the probability generating function for the number of customers in the system and the moments of the system length are obtained. Numerical examples are provided.

Keywords: *Exceptional service, vacation time, steady state probabilities, queue length.*
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Introduction:

Yukata Baba [9] studied queueing systems in which the service time of a first N customer depends on the number of customers served in the current busy period. To mention a few references we would name Welch [8], Li et al [6] and Igaki et al [3].

The single server queueing systems with vacation are analysed by many authors including Levy et al [5], Cooper [1], Doshi [2], Yukata Baba [10] and Teghem [7] have made a comprehensive survey of queueing system with vacation.

In this paper we consider an M/G/1 queueing system in which the first N customer of each busy period receives the exceptional service, in addition the server takes single vacation each time the system becomes empty.

The organisation of the paper is as follows. We describe the model and introduce notation in section 2. In section 3, we obtain the steady state probabilities and the moments of the queue length distribution. The operating characteristics are obtained in section 4 and a numerical study is carried out in section 5 to test the effect of the system performance.

The Model Consider a single server queueing system in which arrival follows Poisson with parameter λ and service time distribution depends on the number of customers served since the beginning of current busy period. The service discipline is FCFS. Let $B_n(x)$ denote the service time distribution function when n customers have been served since the beginning of current busy period.The Laplace-Stieltjes transform (LST) of $B_n(x)$ is defined by $B_n^*(s) = \int_0^\infty e^{-sx} B_n(dx)$. Further more, we assume that the service time distribution becomes stable after some customers have been served in the current busy period. That is, there is a positive integer $N \geq 1$ such that $B_n(x) = B_N(x)$ for $n \geq N.$ Also if there is no customer in the queue the server is allowed to leave the service station for a random period of time whose probability distribution is $V(x)$.

Let $L(t)$ be the number of customers in the system including the one in the server at time $'t'$. Let $M(t)$ be the number of customers served since the beginning of current busy period at time $'t'$. Let $\hat{B}(t)$ be the remaining service time if there are some customers in system at time $'t'$. If the server is on vacation at time $'t', M(t)$ is defined to be zero. Let $\hat{V}(t)$ be the remaining vacation time for the server on vacation.

The queue length distribution at an arbitrary time will be treated by the supplementary variable technique: that is, the joint distribution of the queue length, the number of customers served since the beginning of current busy period, and the remaining service time for the customer receiving service if the server is busy, or the remaining vacation time if the server is on vacation at time $'t'$ form a Markov chain.

Let us define the following notation for our subsequent analysis.

$$P_{i,j}(x,t)dx = P_r\{L(t) = i, M(t) = j, x < \hat{B}(t) \le x + dx\}, (i = 1,2,\dots \& j = 0,1,\dots,N-1)$$

$$P_{i,N}(x,t)dx = P_r\{L(t) = i, M(t) \ge N, x < \hat{B}(t) \le x + dx\}, (i = 1,2,\dots)$$

$$Q_i(x,t)dx = P_r\{L(t) = i, x < \hat{V}(t) \le x + dx\}, \qquad (i = 0,1,2,\dots)$$

$$P^*_{i,j}(s,t) = \int_0^\infty e^{-sx} P_{i,j}(x,t)dx, \qquad i = 1,2,\dots \ and j = 0,1,\dots,N$$

$$Q^*_i(s,t) = \int_0^\infty e^{-sx} Q_i(x,t)dx, \qquad i = 0,1,\dots$$

$$V^*(s) = \int_0^\infty e^{-sx} V(x)dx$$

$$P_{i,j}(x) = \lim_{t\to\infty} P_{i,j}(x,t), \qquad (i = 1,2,\dots; j = 0,\dots,N)$$

$$Q_i(x) = \lim_{t\to\infty} Q_i(x,t), \qquad (i = 0,1,\dots)$$

$$P_{0,0} = \lim_{t\to\infty} P_{0,0}(t)$$

$$\pi^*_j(z,s) = \sum_{i=1}^{\infty} P^*_{i,j}(s) z^i, \qquad (j = 0,\dots,N)$$

$$q_j(z) = \sum_{i=1}^{\infty} P_{i,j}(0) z^i, \qquad (j = 0,\dots,N)$$

$$\psi^*(z,s) = \sum_{i=0}^{\infty} Q^*_i(s) z^i$$

$$\phi(z) = \sum_{i=0}^{\infty} Q_i(0) z^i$$

By assuming that the system is in steady state condition, the differential difference equations governing the system are as follows:

$$\lambda P_{0,0} = Q_0(0) \tag{1}$$

$$\frac{-dQ_0(x)}{dx} = -\lambda Q_0(x) + \sum_{j=0}^{N} P_{1,j}(0)V(x) \tag{2}$$

$$\frac{-dQ_i(x)}{dx} = -\lambda Q_i(x) + \lambda Q_{i-1}(x), \ (i = 1,2,\dots) \tag{3}$$

$$\frac{-dP_{1,0}(x)}{dx} = -\lambda P_{1,0}(x) + \lambda P_{0,0}\frac{B_0(dx)}{dx} + Q_1(0)\frac{B_0(dx)}{dx} \tag{4}$$

$$\frac{-dP_{i,0}(x)}{dx} = -\lambda P_{i,0}(x) + \lambda P_{i-1,0}(x) + Q_i(0)\frac{B_0(dx)}{dx}, \qquad (i = 2,3,\dots) \tag{5}$$

$$\frac{-dP_{1,j}(x)}{dx} = -\lambda P_{1,j}(x) + P_{2,j-1}(0)\frac{B_j(dx)}{dx}, (j = 1,2,\dots,N-1) \tag{6}$$

$$\frac{-dP_{i,j}(x)}{dx} = -\lambda P_{i,j}(x) + \lambda P_{i-1,j}(x) + P_{i+1,j-1}(0)\frac{B_j(dx)}{dx}, (i = 2,3, \dots j = 1,2, \dots, N-1) \quad (7)$$

$$\frac{-dP_{1,N}(x)}{dx} = -\lambda P_{1,N}(x) + \{P_{2,N-1}(0) + P_{2,N}(0)\}\frac{B_N(dx)}{dx} \quad (8)$$

$$\frac{-dP_{i,N}(x)}{dx} = -\lambda P_{i,N}(x) + \lambda P_{i-1,N}(x) + \{P_{i+1,N-1}(0) + P_{i+1,N}(0)\}\frac{B_N(dx)}{dx}, \quad (i = 2,3, \dots) \quad (9)$$

The Analysis:

In this section, we find $Q_0(0)$, the PGF for finding the moments of queue length as well as the stability condition. Taking the LST's of (2)-(9), we have

$$(\lambda - s)Q_0^*(s) = \left[\sum_{j=0}^{N} P_{1,j}(0)\right]V^*(s) - Q_0(0) \quad (10)$$

$$(\lambda - s)Q_i^*(s) = \lambda Q_{i-1}^*(s) - Q_i(0), \ (i = 1,2, \dots) \quad (11)$$

$$(\lambda - s)P_{1,0}^*(s) = [Q_0(0) + Q_1(0)]B_0^*(s) - P_{1,0}(0) \quad (12)$$

$$(\lambda - s)P_{i,0}^*(s) = \lambda P_{i-1,0}^*(s) + Q_i(0)B_0^*(s) - P_{i,0}(0), \ (i = 2,3, \dots) \quad (13)$$

$$(\lambda - s)P_{1,j}^*(s) = P_{2,j-1}(0)B_j^*(s) - P_{1,j}(0), \ (j = 1,2, \dots, N-1) \quad (14)$$

$$(\lambda - s)P_{i,j}^*(s) = \lambda P_{i-1,j}^*(s) + P_{i+1,j-1}(0)B_j^*(s) - P_{i,j} \quad (i = 2,3, \dots; \quad j = 1,2, \dots, N-1) \quad (15)$$

$$(\lambda - s)P_{1,N}^*(s) = \{P_{2,N-1}(0) + P_{2,N}(0)\}B_N^*(s) - P_{1,N}(0) \quad (16)$$

$$(\lambda - s)P_{i,N}^*(s) = \lambda P_{1-1,N}^*(s) + \{P_{i+1,N-1}(0) + P_{i+1,N}(0)\}B_N^*(s) - P_{i,N}(0) \quad (i = 2,3, \dots) \quad (17)$$

Substituting $s = \lambda$ into (10) and (11), we get

$$Q_0(0) = \left[\sum_{j=0}^{N} P_{1,j}(0)\right]V^*(\lambda) \quad (18)$$

$$Q_i(0) = \lambda Q_{i-1}^*(\lambda) \quad (19)$$

Differentiating (10)and (11) $n+1$ times, and inserting $s = \lambda$, we have,

$$-(n+1)Q_0^{*(n)}(\lambda) = \left[\sum_{j=0}^{N} P_{1,j}(0)\right]V^{*(n+1)}(\lambda) \quad (20)$$

$$-(n+1)Q_i^{*(n)}(\lambda) = \lambda Q_{i-1}^{*(n+1)}(\lambda), \ (i = 2,3, \dots) \quad (21)$$

By using recursion in (11)and (10), we have

$$Q_i^*(\lambda) = \frac{(-1)^{i+1}\lambda^i}{(i+1)!}\left[\sum_{j=0}^{N} P_{1,j}(0)\right]V^{*(i+1)}(\lambda), \ (i = 0,1,2, \dots) \quad (22)$$

From (18), (19) and (22), we can express $Q_i(0)$ in terms of $Q_0(0)$,

$$Q_i(0) = \frac{(-1)^i\lambda^i}{i!}\frac{Q_0(0)}{V^*(\lambda)}V^{*(i)}(\lambda); \ (i = 0,1,2, \dots) \quad (23)$$

Substituting $s = \lambda$ in (22) and (23), we get

$$P_{1,0}(0) = [Q_0(0) + Q_1(0)]B_0^*(\lambda) \quad (24)$$

$$P_{i,0}(0) = \lambda P_{i-1,0}^*(\lambda) + Q_i(0)B_0^*(\lambda), (i = 2,3, \dots) \quad (25)$$

Differentiating (22)and (23) $n+1$ times, and inserting $s = \lambda$, we have,

$$-(n+1)P_{1,0}^{*(n)}(\lambda) = [Q_0(0) + Q_1(0)]B_0^{*(n+1)}(\lambda) \quad (26)$$

$$-(n+1)P_{i,0}^{*(n)}(\lambda) = \lambda P_{i-1,0}^{*(n+1)}(\lambda) + Q_i(0)B_0^{*(n+1)}(\lambda), \ (i = 2,3, \dots) \quad (27)$$

By using recursion in (27)and (26), we have

$$P_{i,0}^*(\lambda) = \frac{(-1)^i\lambda^{i-1}}{i!}Q_0(0)B_0^{*(i}(\lambda) + \sum_{k=1}^{i}\frac{(-1)^k(\lambda)^{k-1}}{k!}Q_{i-k+1}(0)B_0^{*(k)}(\lambda), \ (i = 1,2,3, \dots, N) \quad (28)$$

From (24), (25) and (28), we can express $P_{i,0}(0)$ in terms of $Q_0(0)$,

$$P_{i,0}(0) = \frac{(-1)^{i-1}\lambda^{i-1}}{(i-1)!} Q_0(0) B_0^{*(i-1)}(\lambda) + \sum_{k=0}^{i-1} \frac{(-1)^k(\lambda)^k}{k!} Q_{i-k}(0) B_0^{*(k)}(\lambda),\ (i = 1,2,3,\dots,N) \tag{29}$$

Using (23) in (29), we get

$$P_{i,0}(0) = \frac{(-1)^{i-1}\lambda^{i-1}}{(i-1)!} Q_0(0) B_0^{*(i-1)}(\lambda) + \sum_{k=0}^{i-1} \frac{(-1)^i\lambda^i}{k!(i-k)!} \left[\frac{Q_0(0)}{V^*(\lambda)}\right] V^{*(i-k)}(\lambda) B_0^{*(k)}(\lambda),\ (i = 1,2,\dots,N) \tag{30}$$

Substituting $s = \lambda$ in (14) and (15), we get

$$P_{1,j}(0) = P_{2,j-1}(0) B_j^*(\lambda), \qquad (j = 1,2,\dots,N-1) \tag{31}$$

$$P_{i,j}(0) = \lambda P_{i-1,j}^*(\lambda) + P_{i+1,j-1}(0) B_j^*(\lambda),\ (i = 2,3,\dots; \quad j = 1,2,\dots,N-1) \tag{32}$$

Differentiating (14)and (15) $n+1$ times, and inserting $s = \lambda$, we have,

$$-(n+1) P_{1,j}^{*(n)}(\lambda) = P_{2,j-1}(0) B_j^{*(n+1)}(\lambda),\ (j = 1,2,\dots,N-1) \tag{33}$$

$$-(n+1) P_{i,j}^{*(n)}(\lambda) = \lambda P_{i-1,j}^{*(n+1)}(\lambda) + P_{i+1,j-1}(0) B_j^{*(n+1)}(\lambda) \quad (i = 2,3,\dots\ and\ j = 1,2,\dots,N-1) \tag{34}$$

Using (30)-(34) we can calculate $P_{i,j}(0)$, (j=1,2,...,N-1 and i=1,2,...,N-j) in terms of $Q_0(0)$ by the following numerical algorithm.

Algorithm

for $j = 1$ to $N-1$ do
for $i = 1$ to $N-j$ do

$$P_{i,j}(0) = \sum_{k=0}^{i-1} \frac{(-1)^k\lambda^k}{k!} P_{i-k+1,j-1}(0) B_j^{*(k)}(\lambda) \tag{35}$$

Hence $P_{1,j}(0)(j = 1,\dots,N-1)$ can be obtained from (35). Finally, from (18) we get

$$P_{1,N}(0) = \frac{Q_0(0)}{V^*(\lambda)} - \sum_{j=0}^{N-1} P_{1,j}(0) \tag{36}$$

It immediately follows that we can express $P_{1,j}(0)$,(j=1,..., N) in terms of $Q_0(0)$ from (30), (35)and (36).

Multiplying both sides of equation (11) by z^i, summing over i from 1 to ∞ and adding equation (10) to this sum, we get

$$(\lambda - \lambda z - s)\psi^*(z,s) = \{\textstyle\sum_{j=0}^{N} P_{1,j}(0)\} V^*(s) - \phi(z) \tag{37}$$

Multiplying both sides of equation (13) by z^i, summing over i from 2 to ∞, multiplying both sides of equation (12) by z and adding these two equations, we get

$$(\lambda - \lambda z - s)\pi_0^*(z,s) = B_0^*(s)[\phi(z) - Q_0(0)(1-z)] - q_0(z) \tag{38}$$

Multiplying both sides of equation (15) by z^i, summing over i from 2 to ∞, multiplying both sides of equation (14) by z and adding these two equations, we get

$$(\lambda - \lambda z - s)\pi_j^*(z,s) = B_j^*(s)\left[\frac{q_{j-1}(z)}{z} - P_{1,j-1}(0)\right] - q_j(z),\ (j = 1,2,\dots,N-1) \tag{39}$$

Multiplying both sides of equation (17) by z^i, summing over i from 2 to ∞, multiplying both sides of equation (16) by z and adding these two equations, we get

$$(\lambda - \lambda z - s)\pi_N^*(z,s) = B_N^*(s)\{\frac{q_{N-1}(z) + q_N(z)}{z} - P_{1,N-1}(0) - P_{1,N}(0)\} - q_N(z) \tag{40}$$

Substituting $s = \lambda - \lambda z$ in (37)-(40), we get

$$\phi(z) = [\textstyle\sum_{j=0}^{N} P_{1,j}(0)] V^*(\lambda - \lambda z) \tag{41}$$

$$q_0(z) = [\phi(z) - Q_0(0)(1 - z)]B_0^*(\lambda - \lambda z) \quad (42)$$

$$q_j(z) = [\frac{q_{j-1}(z)}{z} - P_{1,j-1}(0)]B_j^*(\lambda - \lambda z),\ (j = 1,2, \dots , N - 1) \quad (43)$$

$$q_N(z) = [\frac{q_{N-1}(z)+q_N(z)}{z} - P_{1,N-1}(0) - P_{1,N}(0)]B_N^*(\lambda - \lambda z) \quad (44)$$

Rearranging (44), we get

$$q_N(z) = \frac{[q_{N-1}(z) - P_{1,N-1}(0)z - P_{1,N}(0)z]B_N^*(\lambda-\lambda z)}{z - B_N^*(\lambda-\lambda z)} \quad (45)$$

Furthermore substituting $s = 0$ in (37)-(40) and using (41)-(44), we get

$$\psi^*(z,0) = \frac{[\sum_{j=0}^{N} P_{1,j}(0)][1 - V^*(\lambda-\lambda z)]}{(\lambda-\lambda z)} \quad (46)$$

$$\pi_0^*(z,0) = \frac{[\phi(z) - Q_0(0)(1-z)][1 - B_0^*(\lambda-\lambda z)]}{(\lambda-\lambda z)} \quad (47)$$

$$\pi_j^*(z,0) = \frac{[q_{j-1}(z) - P_{1,j-1}(0)z][1 - B_j^*(\lambda-\lambda z)]}{z(\lambda-\lambda z)},\ (j = 1,2, \dots , N - 1) \quad (48)$$

$$\pi_N^*(z,0) = \frac{[q_{j-1}(z) + q_N(z) - P_{1,N-1}(0)z - P_{1,N}(0)z][1 - B_N^*(\lambda-\lambda z)]}{z(\lambda-\lambda z)} \quad (49)$$

Substituting $z = 1$ in (41),(42) and (43), we get

$$\phi(1) = \sum_{j=0}^{N} P_{1,j}(0)$$

$$\phi(1) = \frac{Q_0(0)}{V^*(\lambda)} \quad (50)$$

$$q_0(1) = \phi(1) \quad (51)$$

$$q_j(1) = q_{j-1}(1) - P_{1,j-1}(0), \qquad (j = 1, \dots , N - 1) \quad (52)$$

Differentiating (41),(42) and (43), and inserting $z = 1$, we get

$$\phi^{(1)}(1) = [\sum_{j=0}^{N} P_{1,j}(0)]\lambda E(V) \quad (53)$$

$$q_0^{(1)}(1) = [\phi^{(1)}(1) + Q_0(0)] + \phi(1)\lambda E(B_0) \quad (54)$$

$$q_j^{(1)}(1) = q_{j-1}^{(1)}(1) - q_{j-1}(1) + [q_{j-1}(1) - P_{1,j-1}(0)]\lambda E(B_j) \quad (55)$$

Substituting $z = 1$ in (45) and using L'Hospital's rule, we get

$$q_N(1) = \frac{q_{N-1}^{(1)}(1) - P_{1,N-1}(0) - P_{1,N}(0)}{1 - \lambda E(B_N)} \quad (56)$$

Furthermore, substituting $z = 1$ in (46)-(49) and using L'Hospital's rule, we finally obtain,

$$\psi^*(1,0) = [\sum_{j=0}^{N} P_{1,j}(0)]E(V) \quad (57)$$

$$\pi_0^*(1,0) = [\phi(1)]E(B_0) \quad (58)$$

$$\pi_j^*(1,0) = \left[q_{j-1}(1) - P_{1,j-1}(0)\right]E\left(B_j\right), (j = 1,2, \dots , N - 1) \quad (59)$$

$$\pi_N^*(1,0) = [q_{N-1}(1) + q_N(1) - P_{1,N-1}(0) - P_{1,N}(0)]E(B_N) \quad (60)$$

Since $P_{1,j}(0)(j = 0, \dots , N)$ are expressed in terms of $Q_0(0)$, we can express (57)-(60) also in terms of $Q_0(0)$ using (50)-(60).

From the normalizing condition,

$$P_{0,0} + \psi^*(1,0) + \sum_{j=0}^{N} \pi_j^*(1,0) = 1, \quad (61)$$

We get $Q_0(0)$. Therefore, the steady state probabilities immediately follow.

It follows from (56) that this queueing system is stable if and only if $1 - \lambda E(B_N) > 0$, that is $\lambda E(B_N) < 1$.

We define the generating function of the steady state queue length distribution as

$$P(z) = P_{0,0} + \psi^*(z,0) + \sum_{j=0}^{N} \pi_j^*(z,0) \quad (62)$$

Using (46)-(49) into (62), we get

$$P(z) = P_{0,0} + \frac{1}{\lambda(1-z)}\left[\sum_{j=0}^{N} P_{1,j}(0)\right][1 - V^*(\lambda - \lambda z)]$$
$$+\frac{1}{\lambda(1-z)}[\phi(z) - Q_0(0)(1-z)][1 - B_0^*(\lambda - \lambda z)]$$
$$+\sum_{j=1}^{N-1}\frac{1}{\lambda z(1-z)}\left[q_{j-1}(z) - P_{1,j-1}(0)z\right]\left[1 - B_j^*(\lambda - \lambda z)\right]$$
$$+\frac{1}{\lambda z(1-z)}\left[q_{N-1}(z) + q_N(z) - P_{1,N-1}(0)z - P_{1,N}(0)z\right][1 - B_N^*(\lambda - \lambda z)] \quad (63)$$

where $\phi(z)$ and $q_j(z), (j = 0,1, \dots, N)$ are given in equation (41)-(44). $P_{1,j}(0), (j = 0,1, \dots, N)$ given in equation (35).

Particular Cases:

Case (i):

If no customer receives the exceptional service, then (on setting $B_j^*(S) = B^*(S)$; (j=0,1,..., N) in (63)) we have

$$P(z) = \frac{B^*(\lambda-\lambda z)[V^*(\lambda-\lambda z)-1-V^*(\lambda)(1-z)]}{z-B^*(\lambda-\lambda z)}\left[\frac{1-\lambda E(B)}{\lambda E(V)+V^*(\lambda)}\right] \quad (64)$$

Equation (64) is the well known generating function of the steady state queue length distribution of an $M/G/1$ queue with single vacation(Levy and Yechiali(1975)).

Case (ii):

If there is no vacation, then (on setting $V^*(\lambda - \lambda z) = z$ in (63)) we have

$$P(z) = P_{0,0} + \frac{P_{0,0}z[1-B_0^*(\lambda-\lambda z)]}{1-z}$$
$$+\sum_{j=1}^{N-1}\frac{1}{\lambda z(1-z)}\left[q_{j-1}(z) - P_{1,j-1}(0)z\right]\left[1 - B_j^*(\lambda - \lambda z)\right]$$
$$+\frac{1}{\lambda z(1-z)}\left[q_{N-1}(z) + q_N(z) - P_{1,N-1}(0)z - P_{1,N}(0)z\right][1 - B_N^*(\lambda - \lambda z)]$$

Equation (65) is the well known generating function of the steady state queue length distribution of an $M/G/1$ queue with first N customers of each busy period receiving exceptional service(Yukata Baba(1999)).

The Operating Characteristics:

In this section, we derive the average number of customers in the system and the average sojourn time of a customer in this system. Let $E(L)$ denote the mean number of customers in the system and $E(S)$ denote the average sojourn time of a customer in the system.

The average number of customers in the system is

$$E(L) = \frac{d}{dz}[P(z)] \qquad at \;\; z = 1$$
$$= \frac{d}{dz}\left[P_{0,0} + \psi^*(z,0) + \sum_{j=0}^{N}\pi_j^*(z,0)\right]_{z=1}$$
$$= \psi^{*(1)}(1,0) + \sum_{j=0}^{N}\pi_j^{*(1)}(1,0)$$

The average sojourn time of a customer in the system is

$$E(S) = \frac{1}{\lambda}\frac{d}{dz}[P(z)] \qquad at \;\; z = 1$$
$$= \frac{1}{\lambda}[\psi^{*(1)}(1,0) + \sum_{j=0}^{N}\pi_j^{*(1)}(1,0)]$$

Where,

$$\phi(1) = \left[\sum_{j=0}^{N} P_{1,j}(0)\right]$$
$$\phi^{(1)}(1) = \phi(1)\lambda E(V)$$

$$\phi^{(2)}(1) = \phi(1)\lambda^2 E(V)$$
$$q_0(1) = \phi(1)$$
$$q_0^{(1)}(1) = [\phi(1)]\lambda E(B_0) + [\phi^{(1)}(1) + Q_0(0)]$$
$$q_0^{(2)}(1) = [\phi(1)]\lambda^2 E(B_0^2) + 2[\phi^{(1)}(1) + Q_0(0)]\lambda E(B_0) + \phi^{(2)}(1)$$
$$q_j(1) = q_{j-1}(1) - P_{1,j-1}(0)$$
$$q_j^{(1)}(1) = -q_j(1) + \left[q_{j-1}(1) - P_{1,j-1}(0)\right]\lambda E(B_j) + \left[q_{j-1}^{(1)}(1) - P_{1,j-1}(0)\right]$$
$$q_j^{(2)}(1) = -2q_j^{(1)}(1) + \left[q_{j-1}(1) - P_{1,j-1}(0)\right]\lambda^2 E(B_j^2)$$
$$+2[q_{j-1}^{(1)}(1) - P_{1,j-1}(0)]\lambda E(B_j) + q_{j-1}^{(2)}(1)$$
$$q_N(1) = \frac{1}{1-\lambda E(B_N)}\left[q_{N-1}^{(1)}(1) - P_{1,N-1}(0) - P_{1,N}(0)\right]$$
$$q_N^{(1)}(1) = \frac{1}{2[1-\lambda E(B_N)]}[\lambda^2 E(B_N^2)q_N(1)$$
$$+2\left[q_{N-1}^{(1)}(1) - P_{1,N-1}(0) - P_{1,N}(0)\right]\lambda E(B_N) + q_{N-1}^{(2)}(1)]$$
$$\psi^*(1,0) = \left[\sum_{j=0}^{N} P_{1,j}(0)\right]E(V)$$
$$\psi^{*(1)}(1,0) = \frac{1}{2}\left[\sum_{j=0}^{N} P_{1,j}(0)\right]\lambda E(V^2)$$
$$\pi_0^*(1,0) = \phi(1)E(B_0)$$
$$\pi_0^{*(1)}(1,0) = \frac{1}{2}\left[\phi(1)\lambda E(B_0^2) + 2[\phi^{(1)}(1) + Q_0(0)]E(B_0)\right]$$
$$\pi_j^*(1,0) = \left[q_{j-1}(1) - P_{1,j-1}(0)\right]E(B_j)$$
$$\pi_j^{*(1)}(1,0) = \frac{1}{2}\begin{bmatrix}-2\pi_j^*(1,0) + \left[q_{j-1}(1) - P_{1,j-1}(0)\right]\lambda E\left(B_j^2\right) \\ +2[q_{j-1}^{(1)}(1) - P_{1,j-1}(0)]E(B_j)\end{bmatrix}$$
$$\pi_N^*(1,0) = \left[q_{N-1}(1) + q_N(1) - P_{1,N-1}(0) - P_{1,N}(0)\right]E(B_N)$$
$$\pi_N^{*(1)}(1,0) = \frac{1}{2}[-2\pi_N^*(1,0) + [q_{N-1}(1) + q_N(1) - P_{1,N-1}(0) - P_{1,N}(0)]\lambda E(B_N^2)$$
$$+2[q_{N-1}^{(1)}(1) + q_N^{(1)}(1) - P_{1,N-1}(0) - P_{1,N}(0)]E(B_N)]$$

Numerical Study:

In this section, we compare the operating characteristics of our queueing models studied in this paper for $N = 2$ and the $M/G/1$ queue with first $N(N = 2)$ customer of each busy period receiving the exceptional service without server vacation for the arbituary values of system parameters. Let $E(L_1)$ and $E(S_1)$ be the mean of the queue length and the mean sojourn time of a customer of the current model for $N(N = 2)$. Let $E(L_2)$ and $E(S_2)$ be the mean of the queue length and the mean sojourn time of a customer of $M/G/1$ queue with first $N(N = 2)$ customer of each busy period recieving the exceptional service without server vacation. The service time distribution and vacation time ditribution are taken as exponential distributions. That is, $B_0^*(\lambda - \lambda z) = \frac{\mu_0}{\lambda-\lambda z+\mu_0}$, $B_1^*(\lambda - \lambda z) = \frac{\mu_1}{\lambda-\lambda z+\mu_1}$, $B_2^*(\lambda - \lambda z) = \frac{\mu_2}{\lambda-\lambda z+\mu_2}$ and $V^*(\lambda - \lambda z) = \frac{\theta}{\lambda-\lambda z+\theta}$. For $\mu_0 = 2, \mu_1 = 4, \mu_2 = 6, \theta = 8$, and λ varying from 0 to 1,the operating characteristics are calculated and are shown in Fig.1 and Fig.2.

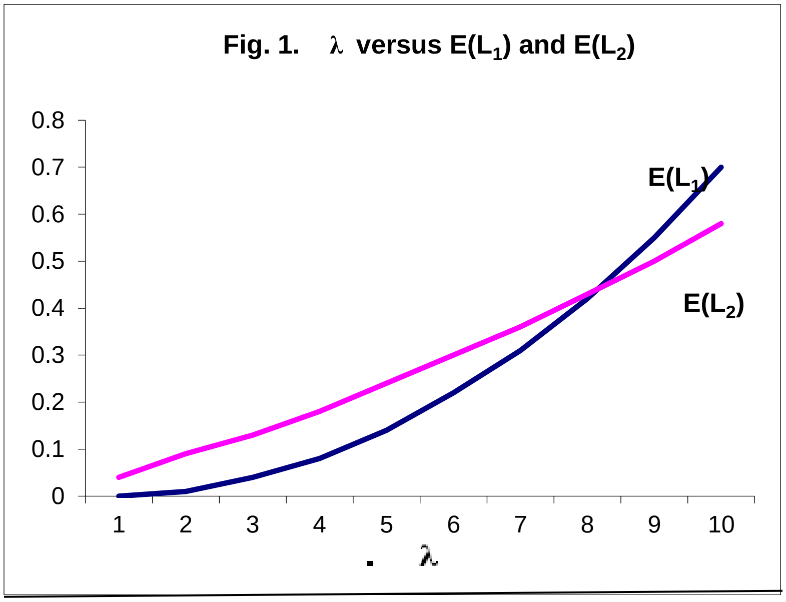

Fig. 1

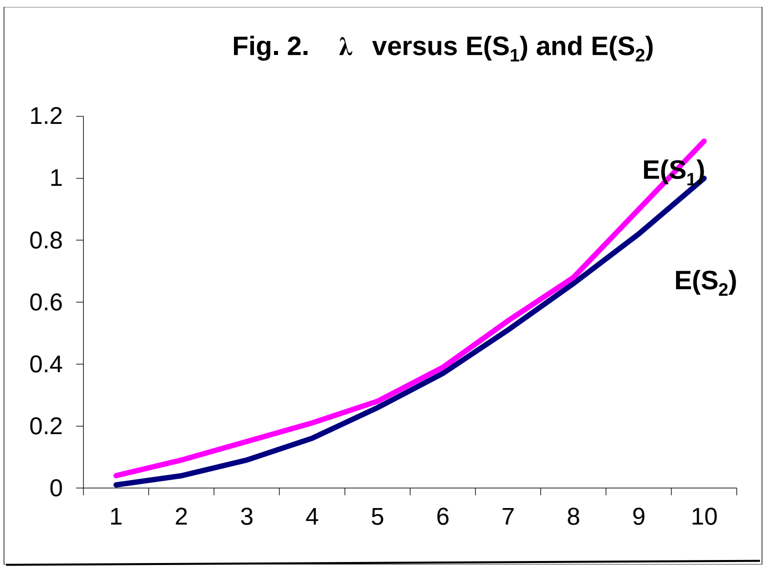

Fig. 2

Figure 1 shows λ versus $E(L_1)$ and $E(L_2)$, figure 2 shows λ versus $E(S_1)$ and $E(S_2)$. From this we conclude that $E(L_1)$ and $E(S_1)$ are larger than $E(L_2)$ and $E(S_2)$.

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